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A probabilistic approach to spectral analysis of growth-fragmentation equations

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The growth-fragmentation equation describes a system of growing and dividing particles, and arises in models of cell division, protein polymerisation and even telecommunications protocols. Several important questions about the equation concern the asymptotic behaviour of solutions at large times: at what rate do they converge to zero or infinity, and what does the asymptotic profile of the solutions look like? Does the rescaled solution converge to its asymptotic profile at an exponential speed? These questions have traditionally been studied using analytic techniques such as entropy methods or splitting of operators. In this work, we present a probabilistic approach to the study of this asymptotic behaviour. We use a Feynman–Kac formula to relate the solution of the growth-fragmentation equation to the semigroup of a Markov process, and characterise the rate of decay or growth in terms of this process. We then identify the spectral radius and the asymptotic profile in terms of a related Markov process, and give a spectral interpretation in terms of the growth-fragmentation operator and its dual. In special cases, we obtain exponential convergence.

Keywords: growth-fragmentation equation, transport equations, cell division equation, one-parameter semigroups, spectral analysis, spectral radius, Feynman–Kac formula, piecewise-deterministic Markov processes, Lévy processes.
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1 Introduction

This work studies the asymptotic behaviour of solutions to the growth-fragmentation equation using probabilistic methods. The growth-fragmentation arises from mathem-

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atical models of biological phenomena such as cell division [29, §4] and protein polymerization [15], as well as in telecommunications [9]. The equation describes the evolution of the density $u_t(x)$ of particles of mass $x > 0$ at time $t \geq 0$, in a system whose dynamics are given as follows. Each particle grows at a certain rate depending on its mass and experiences 'dislocation events', again at a rate depending on its mass. At each such event, it splits into smaller particles in such a way that the total mass is conserved. The growth-fragmentation equation is a partial integro-differential equation and can be expressed in the form

$$\partial_t u_t(x) + \partial_x(c(x)u_t(x)) = \int_x^\infty u_t(y)k(y,x)\mathrm{d}y - K(x)u_t(x), \tag{1}$$

where $c\colon (0,\infty) \to (0,\infty)$ is a continuous positive function specifying the growth rate, $k\colon (0,\infty) \times (0,\infty) \to \mathbb{R}_+$ is a so-called fragmentation kernel, and the initial condition u_0 is prescribed. In words, $k(y,x)$ represents the rate at which a particle with size x appears as the result of the dislocation of a particle with mass $y > x$. More precisely, the fragmentation kernel fulfills

$$k(x,y) = 0 \text{ for } y > x, \text{ and } \int_0^x yk(x,y)\mathrm{d}y = xK(x).$$

The first requirement stipulates that after the dislocation of a particle, only particles with smaller masses can arise. The second reflects the conservation of mass at dislocation events, and gives the interpretation of $K(x)$ as the total rate of dislocation of particles with size x.

This equation has been studied extensively over many years. A good introduction to growth-fragmentation equations and related equations in biology can be found in the monographs of Perthame [29] and Engel and Nagel [12], and a major issue concerns the asymptotic behaviour of solutions u_t. Typically, one wishes to find a constant $\rho \in \mathbb{R}$, the *spectral radius*, for which $\mathrm{e}^{-\rho t}u_t$ converges, in some suitable space, to a so-called *asymptotic profile* v. Ideally, we would also like to have some information about the *rate of convergence*; that is, we would like to find some $r > 0$ with the property that $\mathrm{e}^{-rt}(\mathrm{e}^{-\rho t}u_t - v)$ converges to zero.

For such questions, a key step in finding ρ is the spectral analysis of the growth-fragmentation operator

$$\mathcal{A}f(x) = c(x)f'(x) + \int_0^x f(y)k(x,y)\mathrm{d}y - K(x)f(x), \qquad x > 0, \tag{2}$$

which is defined for smooth compactly supported f, say.

Indeed, observe first that the weak form of the growth-fragmentation equation (1) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u_t, f\rangle = \langle u_t, \mathcal{A}f\rangle, \tag{3}$$

where we use the notation $\langle \mu, g\rangle := \int g(x)\,\mu(\mathrm{d}x)$ for any measure μ and function g on the same space, and $\langle f, g\rangle := \langle \mu, g\rangle$ with $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$ when $f \geq 0$ is a measurable function. Under some simple assumptions that we will specify shortly, there exists a unique semigroup $(T_t)_{t\geq 0}$, defined on a certain Banach space of functions on $(0,\infty)$, whose infinitesimal generator extends $\mathcal{A}$. Then, the solutions u_t of (3) have the representation

$$\langle u_t, f\rangle = \langle u_0, T_t f\rangle.$$

Several authors have shown the existence of a positive eigenfunction associated to the first eigenvalue of the dual operator $\mathcal{A}^*$ and established exponential convergence of the solution to an asymptotic profile, under certain assumptions on c and K. Since the literature is considerable, we refer only to a few works which are quite close to our assumptions or approach: Cáceres et al. [7] study the case of linear growth and K bounded by a power function, via entropy methods; Mischler and Scher [26] use a splitting technique in order to derive a Krein–Rutman theorem, which is effective when c is constant and K is zero in some neighbourhood of 0; and Banasiak et al. [2] study a situation in which particle sizes are bounded, and do so via an interesting connection with stochastic semigroups. Moreover, Calvez et al. [8] investigate the dependence of the leading eigenvalue (i.e. the spectral radius), and the corresponding eigenvector on the coefficients of the equation; and Bouguet [6] studies a conservative version of the equation using a Markov process approach similar to ours.

The purpose of this work is to show the usefulness of stochastic methods in this setting. We have not attempted to find the most general conditions, but rather to demonstrate the benefits of the probabilistic approach. For the sake of simplicity and conciseness, we shall restrict our attention to the case when the growth rate is bounded from above by a linear function, namely

$$\|\underline{c}\|_\infty := \sup_{x>0} c(x)/x < \infty, \tag{4}$$

and we shall shortly make some further technical assumptions on the fragmentation kernel k. We stress that the techniques developed in this work can be adapted to deal with other types of growth and fragmentation rates of interest which have been considered in preceding works.

In short, we will obtain probabilistic representations of the main quantities of interest (the semigroup T_t, the spectral radius ρ, the asymptotic profile v, and so on) in terms of a certain Markov process with values in $(0,\infty)$. Specifically, even though $(T_t)_{t\geq 0}$ is not a Markovian (i.e., contraction) semigroup, the operator

$$\mathcal{G}f(x) := c(x)f'(x) + \int_0^x (f(y) - f(x))\frac{y}{x}k(x,y)\,\mathrm{d}y$$

is the infinitesimal generator of a Markovian semigroup, and this operator is closely connected to $\mathcal{A}$.

To be precise, comparing $\mathcal{A}$ and $\mathcal{G}$ allows us to express the semigroup T_t via a so-called Feynman–Kac formula:

$$T_t f(x) = x\mathbb{E}_x\left(\mathcal{E}_t \frac{f(X_t)}{X_t}\right), \qquad t \geq 0, \quad x > 0, \tag{5}$$

where X is the Markov process with infinitesimal generator $\mathcal{G}$, $\mathbb{P}_x$ and $\mathbb{E}_x$ represent respectively the probability measure and expectation under which X starts at $X_0 = x$, and

$$\mathcal{E}_t \coloneqq \exp\left(\int_0^t \frac{c(X_s)}{X_s}\mathrm{d}s\right), \qquad t \geq 0.$$

Even though the formula (5) is not very explicit in general, we can use it to say quite a lot about the behaviour of T_t as $t \to \infty$.

In this direction, a fundamental role is played by the function $L_{x,y} : \mathbb{R} \to (0, \infty]$ defined as the Laplace transform

$$L_{x,y}(q) \coloneqq \mathbb{E}_x\left(\mathrm{e}^{-qH(y)}\mathcal{E}_{H(y)}, H(y) < \infty\right), \tag{6}$$

where $H(y)$ denotes the first hitting time of y, Indeed, we identify a first quantity of importance in the study of the large time behaviour of $(T_t)_{t\geq 0}$, namely the *spectral radius*, as

$$\rho \coloneqq \inf\left\{q \in \mathbb{R} : L_{x,x}(q) < 1\right\}, \tag{7}$$

where $x > 0$ is arbitrary. The quantity ρ is sometimes called the 'Malthus exponent' in the literature on growth-fragmentation.

Next, we shall focus on the case where

$$L_{x,x}(\rho) = 1 \tag{8}$$

for some (and then all) $x > 0$, and, for arbitrary fixed $x_0 > 0$, set

$$\ell(x) = L_{x,x_0}(\rho).$$

Then, the function

$$x \mapsto \bar{\ell}(x) \coloneqq x\ell(x)$$

can be viewed as an eigenfunction of $\mathcal{A}$ with eigenvalue ρ, whenever the function ℓ is bounded. Furthermore, provided that the function $q \mapsto L_{x,x}(q)$ possesses a finite right-derivative at ρ for some (and then all) $x > 0$, the absolutely continuous measure

$$\nu(\mathrm{d}x) \coloneqq \frac{\mathrm{d}x}{\bar{\ell}(x)c(x)|L'_{x,x}(\rho)|}, \qquad x > 0, \tag{9}$$

is an eigenmeasure of the dual operator $\mathcal{A}^*$, with eigenvalue ρ (at least under some further technical conditions).

Finally, one can describe the asymptotic behaviour of the fragmentation semigroup as follows. For every $x > 0$ and continuous function $f : (0, \infty) \to \mathbb{R}$ with compact support, one has

$$\lim_{t\to\infty} \mathrm{e}^{-\rho t}T_t f(x) = \bar{\ell}(x)\langle \nu, f\rangle. \tag{10}$$

In certain concrete situations, we can furthermore demonstrate exponential convergence, using classical probabilistic techniques.

Technically, the cornerstone of our analysis is that the assumption (8) enables us to define a remarkable martingale multiplicative functional $\mathcal{M}$ of X. In turn, by classical change of probabilities, $\mathcal{M}$ yields another Markov process $(Y_t)_{t\geq 0}$ that is always *recurrent*. Using ergodic theory for recurrent Markov processes then readily leads to the large time asymptotic behaviour of the growth-fragmentation semigroup mentioned above.

The formulas above may seem somewhat cryptic, but could nonetheless be useful in applications, for instance as the basis of a Monte Carlo method for computing the spectral radius and its corresponding eigenfunction and dual eigenmeasure. There are well-established algorithms for efficiently simulating Markov processes, and the process X which appears here falls within the even nicer class of 'piecewise deterministic' Markov processes. This simulation is probably less costly than numerical estimation of the leading eigenvalue and corresponding eigenfunctions of $\mathcal{A}$ and its dual, at least when the spectral gap is small or absent.

The remainder of this article is organised as follows. In section 2, we make precise the relationship between the operators $\mathcal{A}$ and $\mathcal{G}$, and derive the Feynman–Kac formula (5). Along the way, this establishes the existence and uniqueness of solutions to (3). In section 3, we identify the spectral radius ρ and give some simple bounds for this quantity. Under the assumption (8), we give in section 4 a martingale $\mathcal{M}$ for the process X, and apply it in order to show that the function $\bar{\ell}$ is an eigenfunction of $\mathcal{A}$ with eigenvalue ρ. We then use the martingale $\mathcal{M}$, in section 5, to transform X into another Markov process Y, by a classical change of measure. The key point is that the process Y is always recurrent, and this leads to our main result, Theorem 5.3, which comes from the ergodic theory of positive recurrent Markov processes. In this section, we also show that ν is an eigenmeasure of $\mathcal{A}^*$. Finally, in section 6, we specialise our results to the case where the growth rate is linear, that is $c(x) = ax$, and give more explicit results, including criteria for exponential convergence to the asymptotic profile. We also study in some detail a special case where the strongest form of convergence does not hold.

2 Feynman-Kac representation of the semigroup

Our main task in this section is to derive a representation of the semigroup T_t solving the growth-fragmentation equation, using a Feynman–Kac formula. We begin by introducing some notation and listing the assumptions which will be required for our results.

We write $\mathcal{C}_b$ for the Banach space of continuous and bounded functions $f : (0, \infty) \to \mathbb{R}$, endowed with the supremum norm $\|\cdot\|_\infty$. It will be further convenient to set
$\bar{f}(x) = xf(x)$ for every $f \in \mathcal{C}_b$ and $x > 0$, and define $\bar{\mathcal{C}}_b = \{\bar{f} : f \in \mathcal{C}_b\}$. Analogously, we set $\underline{f}(x) = x^{-1}f(x)$.

Recall our assumption (4) that the growth rate c is continuous and is bounded from above by a linear function, that is, in our notation, $\underline{c} \in \mathcal{C}_b$. We further set

$$\bar{k}(x, y) \coloneqq \frac{y}{x} k(x, y),$$

and assume that

$$x \mapsto \bar{k}(x, \cdot) \text{ is a continuous bounded map from } (0, \infty) \text{ to } L^1(\mathrm{d}y). \tag{11}$$

Recall furthermore that the operator $\mathcal{A}$ is defined by (2); in fact, it will be more convenient for us to consider

$$\bar{\mathcal{A}}f(x) = \frac{1}{x}\mathcal{A}\bar{f}(x),$$

which can be written as

$$\bar{\mathcal{A}}f(x) = c(x)f'(x) + \int_0^x (f(y) - f(x))\, \bar{k}(x, y)\mathrm{d}y + \underline{c}(x)f(x). \tag{12}$$

We view $\bar{\mathcal{A}}$ as an operator on $\mathcal{C}_b$ whose domain $\mathcal{D}(\bar{\mathcal{A}})$ contains the space of bounded continuously differentiable functions f such that cf' bounded. Equivalently, $\mathcal{A}$ is seen as an operator on $\bar{\mathcal{C}}_b$ with domain $\mathcal{D}(\mathcal{A}) = \{\bar{f} : f \in \mathcal{D}(\bar{\mathcal{A}})\}$. The following lemma, ensuring the existence and uniqueness of semigroups $\bar{T}_t$ and T_t with infinitesimal generators $\bar{\mathcal{A}}$ and $\mathcal{A}$ respectively, relies on standard arguments.

Lemma 2.1. *Under the assumptions above, we have:*

(i) There exists a unique positive strongly continuous semigroup $(\bar{T}_t)_{t\geq 0}$ on $\mathcal{C}_b$ whose infinitesimal generator coincides with $\bar{\mathcal{A}}$ on the space of bounded continuously differentiable functions f with cf' bounded.

(ii) As a consequence, the identity

$$T_t\bar{f}(x) = x\bar{T}_tf(x), \quad f \in \mathcal{C}_b \text{ and } x > 0$$

defines the unique positive strongly continuous semigroup $(T_t)_{t\geq 0}$ on $\bar{\mathcal{C}}_b$ with infinitesimal generator $\mathcal{A}$.

Proof. Recall that $\underline{c} \in \mathcal{C}_b$ and consider first the operator $\tilde{\mathcal{A}}f := \bar{\mathcal{A}}f - \|\underline{c}\|_\infty f$, that is,

$$\tilde{\mathcal{A}}f(x) = c(x)f'(x) + \int_0^x (f(y) - f(x))\, \bar{k}(x, y)\mathrm{d}y - (\|\underline{c}\|_\infty - \underline{c}(x))f(x),$$

which is defined for f bounded and continuously differentiable with cf' bounded. Plainly $\|\underline{c}\|_\infty - \underline{c} \geq 0$, and we may view $\tilde{\mathcal{A}}$ as the infinitesimal generator of a (sub-stochastic, i.e., killed) Markov process $\tilde{X}$ on $(0, \infty)$. More precisely, it follows from our assumptions (in particular, recall that by (11), the jump kernel $\bar{k}$ is bounded) that the martingale problem for $\tilde{\mathcal{A}}$ is well-posed; this can be shown quite simply using [13, Theorem 8.3.3], for instance. The transition probabilities of $\tilde{X}$ yield a positive contraction semigroup on $\mathcal{C}_b$, say $(\tilde{T}_t)_{t\geq 0}$, that has infinitesimal generator $\tilde{\mathcal{A}}$. Then $\bar{T}_tf := \exp(t\|\underline{c}\|_\infty)\tilde{T}_tf$ defines a positive strongly continuous semigroup on $\mathcal{C}_b$ with infinitesimal generator $\bar{\mathcal{A}}$.

Conversely, if $(\bar{T}_t)_{t\geq 0}$ is a positive strongly continuous semigroup on $\mathcal{C}_b$ with infinitesimal generator $\bar{\mathcal{A}}$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{T}_t\mathbf{1} = \bar{T}_t\bar{\mathcal{A}}\mathbf{1} \leq \|\underline{c}\|_\infty\bar{T}_t\mathbf{1},$$

where $\mathbf{1}$ is the constant function with value 1. It follows that $\|\bar{T}_t f\|_\infty \leq \exp(t\|\underline{c}\|_\infty)\|f\|_\infty$ for all $t \geq 0$ and $f \in \mathcal{C}_b$, and $\tilde{T}_t := \exp(t\|\underline{c}\|_\infty)\bar{T}_t$ defines a positive strongly continuous semigroup on $\mathcal{C}_b$ with infinitesimal generator $\tilde{\mathcal{A}}$. The well-posedness of the martingale problem for $\tilde{\mathcal{A}}$ ensures the uniqueness of $(\tilde{T}_t)_{t\geq 0}$, and thus of $(\bar{T}_t)_{t\geq 0}$.

The second assertion follows from a well-known and easy to check formula for multiplicative transformation of semigroups. $\square$

Although neither $(T_t)_{t\geq 0}$ or $(\bar{T}_t)_{t\geq 0}$ is a contraction semigroup, they both bear a simple relation to a certain Markov process with state space $(0, \infty)$, which we now introduce. The operator

$$\mathcal{G}f(x) := \bar{\mathcal{A}}f(x) - \underline{c}(x)f(x) = c(x)f'(x) + \int_0^x (f(y) - f(x))\bar{k}(x, y)\,\mathrm{d}y, \tag{13}$$

with domain $\mathcal{D}(\mathcal{G}) = \mathcal{D}(\bar{\mathcal{A}})$ is indeed the infinitesimal generator of a conservative (unkilled) Markov process $X = (X_t)_{t\geq 0}$, and in fact, it is easy to check, again using [13, Theorem 8.3.3], that the martingale problem

$$f(X_t) - \int_0^t \mathcal{G}f(X_s)\mathrm{d}s \quad \text{is a martingale for every } \mathcal{C}^1 \text{ function } f \text{ with compact support}$$

is well-posed. In particular, the law of X is characterized by $\mathcal{G}$. We write $\mathbb{P}_x$ for the law of X started from $x > 0$, and $\mathbb{E}_x$ for the corresponding mathematical expectation.

The process X belongs to the class of *piecewise deterministic Markov processes* introduced by Davis [10], meaning that any path $t \mapsto X_t$ follows the deterministic flow $\mathrm{d}x(t) = c(x(t))\mathrm{d}t$, up to a random time at which it makes its first (random) jump. Note further that, since

$$\int_0^1 \frac{\mathrm{d}x}{c(x)} = \int_1^\infty \frac{\mathrm{d}x}{c(x)} = \infty,$$

X can neither enter from 0 nor reach 0 or ∞ in finite time. Finally, it is readily checked that X has the Feller property, in the sense that its transition probabilities depend continuously on the starting point. For the sake of simplicity, we also assume that X is irreducible; this means that, for every starting point $x > 0$, the probability that the Markov process started from x hits a given target point $y > 0$ is strictly positive. Because X is piecewise deterministic and has only downwards jumps, this can be ensured by a simple non-degeneracy assumption on the fragmentation kernel k.

Lemma 2.1(ii) and equation (13) prompt us to consider the exponential functional

$$\mathcal{E}_t := \exp\left(\int_0^t \underline{c}(X_s)\mathrm{d}s\right), \qquad t \geq 0.$$

We note the uniform bound $\mathcal{E}_t \leq \exp(t\|\underline{c}\|_\infty)$, and also observe, from the decomposition of the trajectory of X at its jump times, that there is the identity

$$\mathcal{E}_t = \frac{X_t}{X_0} \prod_{0<s\leq t} \frac{X_{s-}}{X_s}.$$

The point in introducing the elementary transformation and notation above is that it yields a Feynman-Kac representation of the growth-fragmentation semigroup, which appeared as equation (5) in the introduction:

Lemma 2.2. *The growth-fragmentation semigroup $(T_t)_{t\geq 0}$ can be expressed in the form*

$$T_t f(x) = x\mathbb{E}_x\left(\mathcal{E}_t \underline{f}(X_t)\right) = x\mathbb{E}_x\left(\mathcal{E}_t \frac{f(X_t)}{X_t}\right), \qquad f \in \bar{\mathcal{C}}_b.$$

Proof. Recall from Dynkin's formula that for every $f \in \mathcal{D}(\bar{\mathcal{A}})$,

$$f(X_t) - \int_0^t \mathcal{G}f(X_s)\mathrm{d}s\,, \qquad t \geq 0$$

is a $\mathbb{P}_x$-martingale for every $x > 0$. Since $(\mathcal{E}_t)_{t\geq 0}$ is a process of bounded variation with $\mathrm{d}\mathcal{E}_t = \underline{c}(X_t)\mathcal{E}_t\mathrm{d}t$, the integration by parts formula of stochastic calculus [30, Corollary 2 to Theorem II.22] shows that

$$\mathcal{E}_t f(X_t) - \int_0^t \mathcal{E}_s\mathcal{G}f(X_s)\mathrm{d}s - \int_0^t \underline{c}(X_s)\mathcal{E}_s f(X_s)\mathrm{d}s = \mathcal{E}_t f(X_t) - \int_0^t \mathcal{E}_s\bar{\mathcal{A}}f(X_s)\mathrm{d}s$$

is a local martingale. Plainly, this local martingale remains bounded on any finite time interval, and is therefore a true martingale, by [30, Theorem I.51]. We deduce, by taking expectations and using Fubini's theorem, that

$$\mathbb{E}_x(\mathcal{E}_t f(X_t)) - f(x) = \int_0^t \mathbb{E}_x(\mathcal{E}_s\bar{\mathcal{A}}f(X_s))\,\mathrm{d}s$$

holds. Recalling Lemma 2.1(i), this yields the identity $\bar{T}_t f(x) = \mathbb{E}_x(\mathcal{E}_t f(X_t))$, and we conclude the proof with Lemma 2.1(ii). □

We mention that the Feynman-Kac representation of the growth-fragmentation semigroup given in Lemma 2.2 can also be viewed as a 'many-to-one formula' in the setting of branching particle systems (see, for instance, section 1.3 in [35]). Informally, the growth-fragmentation equation describes the evolution of the intensity of a stochastic system of branching particles that grow at rate c and split randomly according to k. In this setting, the Markov process $(X_t)_{t\geq 0}$ with generator $\mathcal{G}$ arises by following the trajectory of a distinguished particle in the system, such that after each dislocation event involving the distinguished particle, the new distinguished particle is selected amongst the new particles according to a size-biased sampling. This particle is referred to as the 'tagged fragment' in certain cases of the growth-fragmentation equation, and we will make this connection more explicit in section 6.

In order to study the long time asymptotic behaviour of the growth-fragmentation semigroup, we seek to understand how $\mathbb{E}_x[\mathcal{E}_t f(X_t)/X_t]$ behaves as $t \to \infty$. We shall tackle this issue in the rest of this work by adapting ideas and techniques of ergodicity for general nonnegative operators, which have been developed mainly in the discrete time setting in the literature; see Nummelin [27] and Seneta [33] for a comprehensive introduction. We shall rely heavily on the fact that the piecewise deterministic Markov process X has no positive jumps, and as a consequence, the probability that the process hits any given single point is positive (points are 'non-polar'.) This enables us to apply the regenerative property of the process at the sequence of times when it returns to its starting point.

3 The spectral radius

Our goal now is to use our knowledge of the Markov process X in order to find the parameter ρ which governs the decay or growth of solutions to the growth-fragmentation equations.

We introduce
$$H(x) \coloneqq \inf\{t > 0 : X_t = x\},$$
the first hitting time of $x > 0$ by X. We stress that, when X starts from $X_0 = x$, $H(x)$ is the first instant (possibly infinite) at which X *returns* for the first time to x. Given $x, y > 0$, the Laplace transform
$$L_{x,y}(q) \coloneqq \mathbb{E}_x\left(\mathrm{e}^{-qH(y)}\mathcal{E}_{H(y)}, H(y) < \infty\right), \qquad q \in \mathbb{R},$$
will play a crucial role in our analysis. We first state a few elementary facts which will be useful in the sequel.

Since X is irreducible, we have $\mathbb{P}_x(H(y) < \infty) > 0$. Moreover, $\mathcal{E}_{H(y)} > 0$ on the event $H(y) < \infty$, from which it follows that $L_{x,y}(q) \in (0, \infty]$. The function $L_{x,y} \colon \mathbb{R} \to (0, \infty]$ is convex, non-increasing, and right-continuous at the boundary point of its domain (by monotone convergence). Furthermore, we have $\mathrm{e}^{-qt}\mathcal{E}_t \leq 1$ for every $q > \|\underline{c}\|_\infty$, and then $L_{x,y}(q) < 1$; indeed,
$$\lim_{q\to-\infty} L_{x,y}(q) = \infty \quad \text{and} \quad \lim_{q\to+\infty} L_{x,y}(q) = 0.$$

The next result is crucial for the identification of the spectral radius.

Proposition 3.1. *Let $q \in \mathbb{R}$ with $L_{x_0,x_0}(q) < 1$ for some $x_0 > 0$. Then $L_{x,x}(q) < 1$ for all $x > 0$.*

Proof. Let $x \neq x_0$ and observe first from the strong Markov property applied at the first hitting time $H(x)$, that
$$\begin{aligned}
1 &> \mathbb{E}_{x_0}(\mathcal{E}_{H(x_0)}\mathrm{e}^{-qH(x_0)}, H(x_0) < \infty) \\
&\geq \mathbb{E}_{x_0}(\mathcal{E}_{H(x_0)}\mathrm{e}^{-qH(x_0)}, H(x) < H(x_0) < \infty) \\
&= \mathbb{E}_{x_0}(\mathcal{E}_{H(x)}\mathrm{e}^{-qH(x)}, H(x) < H(x_0))\mathbb{E}_x(\mathcal{E}_{H(x_0)}\mathrm{e}^{-qH(x_0)}, H(x_0) < \infty) \\
&= \mathbb{E}_{x_0}(\mathcal{E}_{H(x)}\mathrm{e}^{-qH(x)}, H(x) < H(x_0))L_{x,x_0}(q).
\end{aligned}$$

Since $\mathbb{P}_{x_0}(H(x) < H(x_0)) > 0$, because X is irreductible, this entails that

$$0 < \mathbb{E}_{x_0}(\mathcal{E}_{H(x)} \mathrm{e}^{-qH(x)}, H(x) < H(x_0)) < \infty \quad \text{and} \quad 0 < L_{x,x_0}(q) < \infty.$$

Next, we work under $\mathbb{P}_{x_0}$ and write $0 = R_0 < H(x_0) = R_1 < \cdots$ for the sequence of return times at x_0. Using the regeneration at those times, we get

$$\begin{aligned} L_{x_0,x}(q) &= \sum_{n=0}^{\infty} \mathbb{E}_{x_0}(\mathcal{E}_{H(x)} \mathrm{e}^{-qH(x)}, R_n < H(x) < R_{n+1}) \\ &= \sum_{n=0}^{\infty} \mathbb{E}_{x_0}(\mathcal{E}_{R_n} \mathrm{e}^{-qR_n}, R_n < H(x)) \mathbb{E}_{x_0}(\mathcal{E}_{H(x)} \mathrm{e}^{-qH(x)}, H(x) < R_1) \\ &= \mathbb{E}_{x_0}(\mathcal{E}_{H(x)} \mathrm{e}^{-qH(x)}, H(x) < H(x_0)) \sum_{n=0}^{\infty} \mathbb{E}_{x_0}(\mathcal{E}_{H(x_0)} \mathrm{e}^{-qH(x_0)}, H(x_0) < H(x))^n \end{aligned}$$

Plainly,

$$\mathbb{E}_{x_0}(\mathcal{E}_{H(x_0)} \mathrm{e}^{-qH(x_0)}, H(x_0) < H(x)) \le \mathbb{E}_{x_0}(\mathcal{E}_{H(x_0)} \mathrm{e}^{-qH(x_0)}, H(x_0) < \infty) < 1,$$

and summing the geometric series, we get

$$\begin{aligned} L_{x_0,x}(q) &= \frac{\mathbb{E}_{x_0}(\mathcal{E}_{H(x)} \mathrm{e}^{-qH(x)}, H(x) < H(x_0))}{1 - \mathbb{E}_{x_0}(\mathcal{E}_{H(x_0)} \mathrm{e}^{-qH(x_0)}, H(x_0) < H(x))} \\ &< \frac{\mathbb{E}_{x_0}(\mathcal{E}_{H(x)} \mathrm{e}^{-qH(x)}, H(x) < H(x_0))}{\mathbb{E}_{x_0}(\mathcal{E}_{H(x_0)} \mathrm{e}^{-qH(x_0)}, H(x) < H(x_0) < \infty)} = \frac{1}{L_{x,x_0}(q)}, \end{aligned}$$

where the last equality follows from the strong Markov property applied at time $H(x)$ (and we stress that the ratio in the middle is positive and finite.) Hence, we have

$$L_{x_0,x}(q) L_{x,x_0}(q) < 1. \tag{14}$$

We next perform a similar calculation, but now under $\mathbb{P}_x$. Using regeneration at return times at x as above, we see that

$$L_{x,x_0}(q) = \mathbb{E}_x(\mathcal{E}_{H(x_0)} \mathrm{e}^{-qH(x_0)}, H(x_0) < H(x)) \sum_{n=0}^{\infty} \mathbb{E}_x(\mathcal{E}_{H(x)} \mathrm{e}^{-qH(x)}, H(x) < H(x_0))^n.$$

Since we know that $L_{x,x_0}(q) < \infty$, the geometric series above converges, so

$$\mathbb{E}_x(\mathcal{E}_{H(x)} \mathrm{e}^{-qH(x)}, H(x) < H(x_0)) < 1,$$

and

$$L_{x,x_0}(q) = \frac{\mathbb{E}_x(\mathcal{E}_{H(x_0)} \mathrm{e}^{-qH(x_0)}, H(x_0) < H(x))}{1 - \mathbb{E}_x(\mathcal{E}_{H(x)} \mathrm{e}^{-qH(x)}, H(x) < H(x_0))}.$$

Multiplying by $L_{x_0,x}(q)$ and using (14), we deduce that

$$\begin{aligned} 1 - \mathbb{E}_x(\mathcal{E}_{H(x)} \mathrm{e}^{-qH(x)}, H(x) < H(x_0)) &> \mathbb{E}_x(\mathcal{E}_{H(x_0)} \mathrm{e}^{-qH(x_0)}, H(x_0) < H(x)) L_{x_0,x}(q) \\ &= \mathbb{E}_x(\mathcal{E}_{H(x)} \mathrm{e}^{-qH(x)}, H(x_0) < H(x) < \infty), \end{aligned}$$

where again the last equality is seen from the strong Markov property. It follows that $\mathbb{E}_x(\mathcal{E}_{H(x)} \mathrm{e}^{-qH(x)}, H(x) < \infty) = L_{x,x}(q) < 1$. $\square$

We next fix some arbitrary point $x_0 > 0$, and introduce a fundamental quantity.

Definition 3.2. We call

$$\rho \coloneqq \inf\{q \in \mathbb{R} : L_{x_0,x_0}(q) < 1\}$$

the *spectral radius* of the growth-fragmentation operator $\mathcal{A}$.

We stress that Proposition 3.1 shows in particular that the spectral radius ρ does not depend on the choice of x_0. We next justify the terminology by observing that, if $q < \rho$, then

$$\int_0^\infty \mathrm{e}^{-qt} T_t f(x) \mathrm{d}t = \infty$$

for all $x > 0$ and all continuous functions $f \colon (0, \infty) \to \mathbb{R}_+$ with $f \not\equiv 0$, whereas, if $q > \rho$, then there exists a function f which is everywhere positive, and such that

$$\int_0^\infty \mathrm{e}^{-qt} T_t f(x) \mathrm{d}t < \infty$$

for all $x > 0$. The following result actually provides a slightly stronger statement.

Proposition 3.3. *Let $q \in \mathbb{R}$.*

(i) If $L_{x,x}(q) \geq 1$, then for every $f \colon (0, \infty) \to [0, \infty)$ continuous with $f \not\equiv 0$, we have

$$\int_0^\infty \mathrm{e}^{-qt} T_t f(x) \mathrm{d}t = \infty.$$

(ii) If $L_{x,x}(q) < 1$, then there exists a function $f \colon (0, \infty) \to (0, \infty]$ with

$$\lim_{t \to 0} \mathrm{e}^{-qt} T_t f(x) = 0.$$

Proof. (i) Recall from Lemma 2.2 that

$$\int_0^\infty \mathrm{e}^{-qt} T_t f(x) \mathrm{d}t = x \mathbb{E}_x \left(\int_0^\infty \mathrm{e}^{-qt} \mathcal{E}_t \underline{f}(X_t) \mathrm{d}t \right).$$

Decomposing $[0, \infty)$ according to the return times of X at its starting point and applying the regeneration property just as in the proof of Proposition 3.1, we easily find that the quantity above equals

$$x \mathbb{E}_x \left(\int_0^{H(x)} \mathrm{e}^{-qt} \mathcal{E}_t \underline{f}(X_t) \mathrm{d}t \right) \sum_{n=0}^\infty \mathbb{E}_x \left(\mathrm{e}^{-qH(x)} \mathcal{E}_{H(x)}, H(x) < \infty \right)^n.$$

Now the first term above is positive since $f \geq 0$, $f \not\equiv 0$ and X is irreducible, and the series diverges because $\mathbb{E}_x \left(\mathrm{e}^{-qH(x)} \mathcal{E}_{H(x)}, H(x) < \infty \right) = L_{x,x}(q) \geq 1$.

(ii) We take $f(y) = y L_{y,x}(q)$ and observe from the Markov property and Lemma 2.2 that then

$$\mathrm{e}^{-qt} T_t f(x) = x \mathbb{E}_x \left(\mathrm{e}^{-qR(t)} \mathcal{E}_{R(t)}, R(t) < \infty \right),$$

where $R(t)$ denotes the first return time of X to x after time t. We use the notation $\theta_.$ for the usual shift operator; that is, $(X_s, s \geq 0) \circ \theta_t = (X_{s+t}, s \geq 0)$. As before, we denote the sequence of return times of X to its starting point by $R_0 = 0 < R_1 < \cdots$. With this notation, we have that $R(t) = R_{n+1}$ if and only if $R_n \leq t$ and $H(x) \circ \theta_{R_n} > t - R_n$. Regeneration at the return times then enables us to express $\mathrm{e}^{-qt} T_t f(x)$ as

$$
\begin{aligned}
& x \sum_{n=0}^{\infty} \int_{[0,t]} \mathbb{E}_x \left(\mathrm{e}^{-qR_n} \mathcal{E}_{R_n}, R_n \in \mathrm{d}s \right) \mathbb{E}_x \left(\mathrm{e}^{-qH(x)} \mathcal{E}_{H(x)}, t - s < H(x) < \infty \right) \\
& =: x \int_{[0,t]} U^q(x, \mathrm{d}s) \varphi_x(t - s),
\end{aligned}
$$

On the one hand, we observe, again by regeneration, that the total mass of the measure $U^q(x, \cdot)$ is given by

$$
U^q(x, [0, \infty)) = \sum_{n=0}^{\infty} \mathbb{E}_x \left(\mathrm{e}^{-qR_n} \mathcal{E}_{R_n}, R_n < \infty \right) = \sum_{n=0}^{\infty} L_{x,x}(q)^n < \infty,
$$

On the other hand, since

$$
\mathbb{E}_x \left(\mathrm{e}^{-qH(x)} \mathcal{E}_{H(x)}, H(x) < \infty \right) = L_{x,x}(q) < \infty,
$$

we know that $\lim_{t \to \infty} \varphi_x(t) = 0$. Hence, for every $s \geq 0$, we have $\lim_{t \to \infty} \varphi_x(t - s) = 0$, and since $0 \leq \varphi_x(t - s) \leq L_{x,x}(q)$ and the measure $U^q(x, \cdot)$ is finite, we can conclude the proof by dominated convergence. □

We now conclude this section by describing the following elementary bounds for the spectral radius.

Proposition 3.4. *(i) It always holds that $\rho \leq \|\underline{c}\|_\infty$.*

(ii) It holds that $\rho > 0$ whenever X is recurrent; furthermore, if X is positive recurrent with stationary law π, then

$$
\rho \geq \langle \pi, \underline{c} \rangle.
$$

Proof. (i) This follows from the elementary observations preceding Definition 3.2.

(ii) If X is recurrent, then $\mathbb{P}_x(H(x) < \infty) = 1$ and $L_{x,x}(0) = \mathbb{E}_x(\mathcal{E}_{H(x)}) \in (1, \infty]$. This forces $\rho > 0$, since $L_{x,x}(\rho) \leq 1$ by right-continuity of $L_{x,x}$. Furthermore, we may apply the regeneration property at the n-th return time of X to x_0, say R_n, and observe that

$$
\mathbb{E}_{x_0} \left(\mathrm{e}^{-qR_n} \mathcal{E}_{R_n} \right) = L_{x_0,x_0}(q)^n
$$

converges to 0 as $n \to \infty$ for every $q > \rho$. By the ergodic theorem for positive recurrent Markov processes [20, Theorem 20.20],

$$
\ln \mathcal{E}_{R_n} = \int_0^{R_n} \underline{c}(X_s) \mathrm{d}s \sim \langle \pi, \underline{c} \rangle R_n \quad \text{as } n \to \infty, \quad \mathbb{P}_{x_0}\text{-a.s.},
$$

and we then see from Fatou's Lemma that $\lim_{n \to \infty} \mathbb{E}_{x_0} \left(\mathrm{e}^{-qR_n} \mathcal{E}_{R_n} \right) = \infty$, as long as $q < \langle \pi, \underline{c} \rangle$. This entails our last claim. □

4 A martingale multiplicative functional

In short, the purpose of this section is to construct a remarkable martingale which we will then use to transform the Markov process X. We shall obtain a recurrent Markov process Y which in turn will enable us to reduce the analysis of the asymptotic behaviour of T_t to results from ergodic theory. This requires the following assumption to hold:

$$L_{x_0,x_0}(\rho) = 1. \tag{15}$$

Note that, by the right-continuity of $L_{x,x}$, we always have $L_{x_0,x_0}(\rho) \le 1$.

We start with some simple observations relating (15) to the value of L_{x_0,x_0} at the left endpoint of its domain.

Lemma 4.1. *Define* $q_* := \inf\{q \in \mathbb{R} : L_{x_0,x_0}(q) < \infty\}$. *Then:*

(i) Condition (15) *holds if and only if* $L_{x_0,x_0}(q_*) \in [1, \infty]$.

(ii) If $L_{x_0,x_0}(q_*) \in (1, \infty]$, *then* L_{x_0,x_0} *possesses a finite right-derivative at* ρ *and*

$$\mathbb{E}_{x_0}\left(H(x_0)\mathrm{e}^{-\rho H(x_0)}\mathcal{E}_{H(x_0)}, H(x_0) < \infty\right) = -L'_{x_0,x_0}(\rho) < \infty.$$

Proof. Recall that $q_* \le \|\underline{c}\|_\infty$ and that L_{x_0,x_0} is convex and decreasing. We have

$$\lim_{q\to\infty} L_{x_0,x_0}(q) = 0 \quad \text{and} \quad \lim_{q\to q_*+} L_{x_0,x_0}(q) = L_{x_0,x_0}(q_*)$$

by dominated convergence for the first limit, and by monotone convergence for the second. This yields our first claim. For the second, it suffices to observe that if $L_{x_0,x_0}(q_*) > 1$, then $\rho > q_*$ and thus, by convexity, the right derivative of L_{x_0,x_0} at ρ is finite. □

We assume throughout the rest of this section that (15) holds, and describe some remarkable properties of the function $(x, y) \mapsto L_{x,y}(\rho)$ which follow from this assumption.

Lemma 4.2. *Assume that* (15) *holds for some* $x_0 > 0$. *Then*

(i) $L_{x,x}(\rho) = 1$ *for all* $x > 0$, *i.e.,* (15) *actually holds with* x_0 *replaced by any* $x > 0$.

(ii) For all $x, y > 0$, *we have*

$$L_{x,y}(\rho)L_{y,x}(\rho) = 1.$$

(iii) For all $x, y, z > 0$, *there is the identity*

$$L_{x,y}(\rho)L_{y,z}(\rho) = L_{x,z}(\rho).$$

Proof. (i) Indeed, the strict inequality $L_{x,x}(\rho) < 1$ is ruled out by Proposition 3.1. On the other hand, we always have $L_{x,x}(\rho) \le 1$ by the right-continuity of $L_{x,x}$, since, again by Proposition 3.1, $\rho = \inf\{q \in \mathbb{R} : L_{x,x}(q) < 1\}$.

(ii) Using the regeneration at return times at x just as in the proof of Proposition 3.1, we easily get

$$
\begin{aligned}
L_{x,y}(\rho) &= \frac{\mathbb{E}_x(\mathcal{E}_{H(y)}\mathrm{e}^{-\rho H(y)}, H(y) < H(x))}{1 - \mathbb{E}_x(\mathcal{E}_{H(x)}\mathrm{e}^{-\rho H(x)}, H(x) < H(y))} \\
&= \frac{\mathbb{E}_x(\mathcal{E}_{H(y)}\mathrm{e}^{-\rho H(y)}, H(y) < H(x))}{\mathbb{E}_x(\mathcal{E}_{H(x)}\mathrm{e}^{-\rho H(x)}, H(y) < H(x) < \infty)} = \frac{1}{L_{y,x}(\rho)},
\end{aligned}
$$

where the last equality follows from the strong Markov property applied at time $H(y)$.

(iii) Finally, recall that X has no positive jumps, so for every $x < y < z$, we have $H(y) < H(z)$, $\mathbb{P}_x$-a.s. on the event $H(z) < \infty$, and the strong Markov property readily yields (iii) in that case. Using (ii), it is then easy to deduce that (iii) holds in full generality, no matter the relative positions of x, y and z. ☐

Corollary 4.3. *The function $(x, y) \mapsto L_{x,y}(\rho)$ is continuous on $(0, \infty)$ in each of the variables x and y.*

Proof. We only need to check that $\lim_{y\to x} L_{x,y}(\rho) = 1$. If this holds, then Lemma 4.2(iii) then entails the continuity of $z \mapsto L_{x,z}(\rho)$ and we can conclude from Lemma 4.2(ii) that $x \mapsto L_{x,y}(\rho)$ is also continuous.

In this direction, observe first that X has no positive jumps and follows a positive flow velocity between its jump times. Thus, $\mathbb{P}_x$-a.s., on the event $H(x) < \infty$, there exists a unique instant $J \in (0, H(x))$ such that $X_t > x$ for $0 < t < J$ and $X_t < x$ for $J < t < H(x)$. Further, X is continuous at times 0 and $H(x)$. In particular, we have $\mathbb{P}_x$-a.s. that $\lim_{y\to x+} H(y) = 0$ whereas $\lim_{y\to x-} H(y) = H(x)$, and actually, the following limits

$$
\begin{aligned}
&\lim_{y\to x+} \mathrm{e}^{-\rho H(y)}\mathcal{E}_{H(y)}\mathbf{1}_{\{H(y)<\infty\}} = 1, \\
&\lim_{y\to x-} \mathrm{e}^{-\rho H(y)}\mathcal{E}_{H(y)}\mathbf{1}_{\{H(y)<\infty\}} = \mathrm{e}^{-\rho H(x)}\mathcal{E}_{H(x)}\mathbf{1}_{\{H(x)<\infty\}},
\end{aligned}
$$

hold $\mathbb{P}_x$-a.s. We observe that the $\mathbb{P}_x$-expectation of the last quantity is $L_{x,x}(\rho) = 1$ (by Lemma 4.2(i)), and deduce from Fatou's lemma that

$$
\liminf_{y\to x} L_{x,y}(\rho) \geq 1.
$$

On the other hand, recall that $K(x) = \int_0^x \bar{k}(x, y)\mathrm{d}y$ is the total rate of jumps at location x. An easy consequence of the fact that X follows the flow velocity given by $\mathrm{d}x(t) = c(x(t))\mathrm{d}t$ between its jumps, is that the probability under $\mathbb{P}_y$ of the event Λ_x that X has no jump before hitting $x > y$ is given by

$$
\mathbb{P}_y(\Lambda_x) = \exp\left(-\int_y^x \frac{K(z)}{c(z)}\mathrm{d}z\right),
$$

a quantity which converges to 1 as $y \to x-$. Moreover, the time $h(x)$ at which the flow velocity started from y reaches the point x is given by

$$h(y,x) = \int_y^x \frac{1}{c(s)}\,\mathrm{d}s,$$

a quantity which converges to 0 as $y \to x-$. Using $L_{y,x}(\rho) \geq \mathrm{e}^{-\rho h(y,x)}\mathbb{P}_y(\Lambda_x)$, we deduce that $\liminf_{y\to x-} L_{y,x}(\rho) \geq 1$, and then, thanks to Lemma 4.2(ii) that

$$\limsup_{y\to x-} L_{x,y}(\rho) \leq 1,$$

from which it follows that $\lim_{y\to x-} L_{x,y}(\rho) = 1$ and, by the Lemma 4.2(iii), that also $\lim_{y\to x-} L_{y,x}(\rho) = 1$.

Finally, working now under $\mathbb{P}_x$ and, just as above, denoting by Λ_y the event that X makes no jumps before hitting y, we obtain by monotone convergence that

$$\lim_{y\to x+} \mathbb{E}_x\left[\mathrm{e}^{-\rho H(x)}\mathcal{E}_{H(x)}\mathbb{1}_{\Lambda_y}\,\mathbb{1}_{\{H(x)<\infty\}}\right] = L_{x,x}(\rho) = 1.$$

If we write $h(x,y)$ for the hitting time of y by the flow velocity $x(\cdot)$ started from x, and observe that $\int_0^{h(x,y)} \underline{c}(x(s))\mathrm{d}s = \ln(y/x)$, we obtain by the Markov property at time $h(x,y)$ that

$$\mathbb{E}_x\left[\mathrm{e}^{-\rho H(x)}\mathcal{E}_{H(x)}\mathbb{1}_{\Lambda_y}\mathbb{1}_{\{H(x)<\infty\}}\right] = \mathrm{e}^{-\rho h(x,y)}\frac{y}{x}L_{y,x}(\rho).$$

Since $\lim_{y\to x+} h(x,y) = 0$, we conclude, using again Lemma 4.2(ii) for the second equality below, that

$$\lim_{y\to x+} L_{y,x}(\rho) = 1 = \lim_{y\to x+} L_{x,y}(\rho),$$

and the proof is complete. $\square$

Once again, we recall our standing assumption that (15) holds. The following function will be crucial for our analysis:

$$\ell(x) = L_{x,x_0}(\rho)\,, \qquad x > 0.$$

Note from Lemma 4.2(iii) that, for any $y_0 > 0$ and $x > 0$, $L_{x,y_0}(\rho) = \ell(x)L_{x_0,y_0}(\rho)$, and so replacing x_0 by y_0 would only affect the function ℓ by a constant factor. Further, we know from Corollary 4.3 that ℓ is continuous and positive on $(0,\infty)$; in particular, it remains bounded away from 0 and from ∞ on compact subsets of $(0,\infty)$.

We then introduce the multiplicative functional

$$\mathcal{M}_t := \mathrm{e}^{-\rho t}\mathcal{E}_t\frac{\ell(X_t)}{\ell(X_0)}\,, \qquad t \geq 0.$$

The qualifier *multiplicative* stems from the identity $\mathcal{M}_{t+s} = \mathcal{M}_s \circ \theta_t \times \mathcal{M}_t$, where θ_t denotes the usual shift operator. Our strategy in the sequel shall be to make a change of measure with respect to this multiplicative functional. The following result is therefore very important for our goal.

Theorem 4.4. *For every $x > 0$, the multiplicative functional $(\mathcal{M}_t)_{t\geq 0}$ is a $\mathbb{P}_x$-martingale with respect to the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of X.*

Proof. Without loss of generality, we shall work under $\mathbb{P}_{x_0}$. We also define the random variables $R_0 = 0 < R_1 \coloneqq H(x_0) < R_2 < \cdots$ to be the sequence of return times to the point x_0, and recall from the regenerative property at these return times that for every $n \geq 0$, conditionally on $R_n < \infty$, the ratio

$$\frac{\mathrm{e}^{-\rho R_{n+1}}\mathcal{E}_{R_{n+1}}}{\mathrm{e}^{-\rho R_n}\mathcal{E}_{R_n}} = \exp\left(\int_{R_n}^{R_{n+1}} (\underline{c}(X_s) - \rho)\mathrm{d}s\right)$$

is independent of $\mathcal{F}_{R_n}$ and has the same law as $\mathcal{E}_{H(x_0)}\mathrm{e}^{-\rho H(x_0)}$ under $\mathbb{P}_{x_0}$. We see from (15) that $\mathbb{E}_{x_0}\left(\mathcal{E}_{R_n}\mathrm{e}^{-\rho R_n}, R_n < \infty\right) = 1$ for every $n \geq 0$, and it then follows from the Markov property that there is the identity

$$\begin{aligned} \mathbb{E}_{x_0}\left(\mathcal{M}_{R_n}, R_n < \infty \mid \mathcal{F}_t\right) &= \mathbb{E}_{x_0}\left(\mathrm{e}^{-\rho R_n}\mathcal{E}_{R_n}, R_n < \infty \mid \mathcal{F}_t\right) \\ &= \mathrm{e}^{-\rho(t\wedge R_n)}\mathcal{E}_{t\wedge R_n}\ell(X_{t\wedge R_n}) \\ &= \mathcal{M}_{t\wedge R_n}. \end{aligned}$$

As a consequence, the stopped process $(\mathcal{M}_{t\wedge R_n})_{t\geq 0}$ is a martingale.

Further, if we introduce the tilted probability measure

$$\mathbb{Q}^n = \mathbf{1}_{R_n<\infty}\mathrm{e}^{-\rho R_n}\mathcal{E}_{R_n}\mathbb{P}_{x_0} = \mathbf{1}_{R_n<\infty}\mathcal{M}_{R_n}\mathbb{P}_{x_0},$$

then we see by the regeneration property at the return times and the fact that $\mathcal{M}$ is a multiplicative functional, that under $\mathbb{Q}^n$, the variables $R_1, R_2 - R_1, \ldots, R_n - R_{n-1}$ are i.i.d. with law

$$\mathbb{Q}^n(H(x_0) \in \mathrm{d}s) = \mathbb{P}_{x_0}(\mathrm{e}^{-\rho H(x_0)}\mathcal{E}_{H(x_0)}, H(x_0) \in \mathrm{d}s)\,, \qquad s \in (0, \infty).$$

We stress that this distribution does not depend on n, and in particular, for every $t > 0$, we have

$$\mathbb{E}_{x_0}\left(\mathcal{M}_{R_n}, R_n \leq t\right) = \mathbb{Q}^n(R_n \leq t) \longrightarrow 0 \text{ as } n \to \infty.$$

To complete the proof, it now suffices to write for every $t \geq s \geq 0$

$$\begin{aligned} \mathcal{M}_{s\wedge R_n} &= \mathbb{E}_{x_0}(\mathcal{M}_{t\wedge R_n} \mid \mathcal{F}_s) \\ &= \mathbb{E}_{x_0}(\mathcal{M}_t, R_n > t \mid \mathcal{F}_s) + \mathbb{E}_{x_0}(\mathcal{M}_{R_n}, R_n \leq t \mid \mathcal{F}_s), \end{aligned}$$

and we conclude by letting $n \to \infty$ that $\mathcal{M}_s = \mathbb{E}_{x_0}(\mathcal{M}_t \mid \mathcal{F}_s)$. □

We point out that the continuity of ℓ (which is a special case of Corollary 4.3) could also be established from Theorem 4.4 and classical regularity properties of martingales. We conclude this section by the following easy consequence of Theorem 4.4. Under rather mild assumptions, we identify the function $\bar{\ell}(x) = x\ell(x)$ as an eigenfunction of the growth-fragmentation operator $\mathcal{A}$, with eigenvalue given by the spectral radius ρ.

Corollary 4.5. *(i) The function ℓ belongs to the extended domain of the infinitesimal generator $\mathcal{G}$ of X with $\mathcal{G}\ell = (\rho - \underline{c})\ell$, in the sense that the process*

$$\ell(X_t) - \int_0^t (\rho - \underline{c}(X_s))\, \ell(X_s)\mathrm{d}s \tag{16}$$

is a martingale under $\mathbb{P}_x$ for every $x > 0$.

(ii) If ℓ is bounded on $(0, \infty)$, then $\bar{\ell} \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A}\bar{\ell} = \rho\bar{\ell}$.

Proof of Corollary 4.5. (i) Indeed, it suffices to write

$$\ell(X_t) = \ell(x)\mathcal{M}_t \mathrm{e}^{\rho t} \exp\left(-\int_0^t \underline{c}(X_s)\mathrm{d}s\right)$$

and apply stochastic integration by parts. We obtain

$$\ell(X_t) = \ell(x) + \ell(x)\int_0^t \mathrm{e}^{\rho s}\mathcal{E}_s \mathrm{d}\mathcal{M}_s + \int_0^t (\rho - \underline{c}(X_s))\, \ell(X_s)\mathrm{d}s.$$

On the time interval $[0, t]$, the integrand $\mathrm{e}^{\rho s}\mathcal{E}_s$ in the stochastic integral is bounded by a constant, and this entails that the process in (16) is a martingale, by [30, Theorem I.51].

(ii) Recall that we already know that ℓ is continuous, so if further ℓ is bounded, then $\ell \in \mathcal{C}_b$. Then also $(\rho - \underline{c})\ell \in \mathcal{C}_b$, and, by taking expectations in (16) and using the Feller property of X, (i) entails that ℓ belongs to the domain of the infinitesimal generator $\mathcal{G}$, that is $\ell \in \mathcal{D}(\bar{\mathcal{A}})$ or equivalently $\bar{\ell} \in \mathcal{D}(\mathcal{A})$, with $\mathcal{G}\ell = (\rho - \underline{c})\ell$. Since $\mathcal{G}f(x) = x^{-1}\mathcal{A}\bar{f}(x) - \underline{c}(x)f(x)$, we conclude that $\mathcal{A}(\bar{\ell}) = \rho\bar{\ell}$. □

In order to apply Corollary 4.5(ii), we need explicit conditions ensuring that ℓ is bounded, and in this direction we record the following result.

Lemma 4.6. *Assume that*

$$\limsup_{x\to 0+} \underline{c}(x) < \rho \quad \textit{and} \quad \limsup_{x\to\infty} \underline{c}(x) < \rho.$$

Then $\ell \in \mathcal{C}_b$.

Proof. Under the assumptions of the statement, there exists $\rho' < \rho$ such that the set $\{x > 0 : \underline{c}(x) \geq \rho'\}$ is a compact subset of $(0, \infty)$; assume that it is contained in $[a, b]$, for some $0 < a < x_0 < b$. Now, since ℓ is continuous, it is certainly bounded on $[a, b]$. Moreover, if $0 < x < a$, then $\mathrm{e}^{-\rho H(a)}\mathcal{E}_{H(a)} \leq \mathrm{e}^{-(\rho-\rho')H(a)} \leq 1$. So $L_{x,a}(\rho) \leq 1$, and by Lemma 4.2(iii), ℓ remains bounded on $(0, a)$.

Similarly, if now $x > b$ and $H(a, b) := \inf\{t > 0 : X_t \in [a, b]\}$ denotes the first entrance time in $[a, b]$, then again $\mathrm{e}^{-\rho H(a,b)}\mathcal{E}_{H(a,b)} \leq \mathrm{e}^{-(\rho-\rho')H(a)} \leq 1$. By the strong Markov property applied at time $H(a, b)$, we conclude that $\ell(x) \leq \max_{[a,b]} \ell$, so ℓ remains bounded on (b, ∞). □

5 Applying ergodic theory for Markov processes

We still assume that (15) holds throughout this section. Having established the existence of the martingale multiplicative functional $\mathcal{M}$, we use this to 'tilt' the initial probability measure $\mathbb{P}_x$. In other words, we introduce a new probability measure $\mathbb{Q}_x$, defined by the following formula for every $A \in \mathcal{F}_t$:

$$\mathbb{Q}_x(A) = \mathbb{E}_x[\mathbb{1}_A \mathcal{M}_t].$$

Since $\mathbb{P}_x$ is a probability law on the space of càdlàg paths, the same holds for $\mathbb{Q}_x$; and it is convenient to denote by $Y = (Y_t)_{t\geq 0}$ a process with distribution $\mathbb{Q}_x$. For clarity, let us point out that its finite-dimensional distributions are given as follows. Let $0 \leq t_1 < \cdots < t_n \leq t$, and $F\colon \mathbb{R}^n \to \mathbb{R}$. Then

$$\mathbb{Q}_x[F(Y_{t_1}, \dots, Y_{t_n})] = \mathbb{E}_x[\mathcal{M}_t F(X_{t_1}, \dots, X_{t_n})], \qquad x > 0.$$

(Note that, whenever it will not cause confusion, we will use $\mathbb{Q}_x$ not just for the probability measure, but also for expectations under this measure.) In fact, Y is not just a stochastic process, but a Markov process, and we can specify its distribution in detail, as follows.

Lemma 5.1. *Let $x > 0$.*

(i) Under the measure $\mathbb{Q}_x$, $Y = (Y_t)_{t\geq 0}$ is a strong Markov process. The domain of its extended infinitesimal generator $\mathcal{G}_Y$ contains $\mathcal{D}_\ell(\mathcal{G}) \coloneqq \{g : g\ell \in \mathcal{D}(\mathcal{G})\}$, and is given by

$$\mathcal{G}_Y g(x) = \frac{1}{\ell(x)} \mathcal{G}(g\ell)(x) + (\underline{c}(x) - \rho) g(x) \tag{17}$$

in the sense that, for every $x > 0$ and $g \in \mathcal{D}_\ell(\mathcal{G})$,

$$g(Y_t) - \int_0^t \mathcal{G}_Y g(Y_s)\,\mathrm{d}s \quad \textit{is a local martingale under } \mathbb{Q}_x. \tag{18}$$

Its semigroup $(T_t^Y)_{t\geq 0}$, defined on the Banach space

$$\mathcal{C}_b^\ell \coloneqq \{g : (0,\infty) \to (0,\infty) : g\ell \in \mathcal{C}_b\}$$

with norm $\|g\| = \|g\ell\|_\infty$, is given by

$$T_t^Y g(x) \coloneqq \mathbb{Q}_x[g(Y_t)] = \mathbb{E}_x(\mathcal{M}_t g(X_t)) = \frac{1}{\ell(x)} \mathbb{E}_x\left(\mathrm{e}^{-\rho t} \mathcal{E}_t \ell(X_t) g(X_t)\right).$$

(ii) Y is point recurrent.

Proof. (i) It is well-known that transformations based on multiplicative functionals preserve the (strong) Markov property; we refer to [31, §III.19] for a readable account of a slightly simpler case, or [34, §62] for a technical discussion. We can thus view $\mathbb{Q}_x$ as the law of a Markov process $(Y_t)_{t\geq 0}$ with values in $(0,\infty)$, whose semigroup is given by T_t^Y.

We now prove (18) for every $x > 0$. Indeed, we know that $f(X_t) - \int_0^t \mathcal{G}f(X_s)\mathrm{d}s$ is a $\mathbb{P}_x$-martingale, so by stochastic calculus,

$$\mathrm{e}^{-\rho t}\mathcal{E}_t f(X_t) - \int_0^t \mathrm{e}^{-\rho s}\mathcal{E}_s \left(\mathcal{G}f(X_s) + (\underline{c}(X_s) - \rho)f(X_s)\right)\mathrm{d}s$$

is a $\mathbb{P}_x$-local martingale. Multiplying by $\ell(x)$, this shows that

$$\mathcal{M}_t g(X_t) - \int_0^t \frac{\mathcal{M}_s}{\ell(X_s)} \left(\mathcal{G}f(X_s) + (\underline{c}(X_s) - \rho)f(X_s)\right)\mathrm{d}s$$

is a $\mathbb{P}_x$-local martingale. Further, since $\mathcal{M}$ is a $\mathbb{P}_x$-martingale, stochastic integration by parts shows that for every locally bounded function h,

$$\mathcal{M}_t \int_0^t h(X_s)\mathrm{d}s - \int_0^t \mathcal{M}_s h(X_s)\mathrm{d}s$$

is again $\mathbb{P}_x$-local martingale. Putting the pieces together, we get that

$$\mathcal{M}_t \left(g(X_t) - \int_0^t \frac{\mathcal{G}f(X_s) + (\underline{c}(X_s) - \rho)f(X_s)}{\ell(X_s)}\mathrm{d}s \right)$$

is a $\mathbb{P}_x$-local martingale, that is, equivalently, (18) holds.

(ii) Write $H_Y(x) = \inf\{t > 0 : Y_t = x\}$ for first hitting time of x by the process Y. Then:

$$\begin{aligned}
\mathbb{Q}_x(H_Y(x) < \infty) &= \lim_{t\to\infty} \mathbb{Q}_x(H_Y(x) \le t) \\
&= \lim_{t\to\infty} \mathbb{E}_x\left(\mathcal{M}_t, H(x) \le t\right) \\
&= \lim_{t\to\infty} \mathbb{E}_x\left(\mathcal{M}_{H(x)}, H(x) \le t\right) \\
&= \mathbb{E}_x\left(\mathcal{M}_{H(x)}, H(x) < \infty\right) = 1,
\end{aligned}$$

where at the third equality, we used the optional sampling theorem [31, Theorem II.77.5] for the martingale $\mathcal{M}$. □

We next specify classical formulas for invariant measures and stationary distributions of point-recurrent Markov processes, in the case of the process Y.

Corollary 5.2. *(i) The occupation measure m_0 of the excursion of Y away from x_0 defined by*

$$\langle m_0, f\rangle := \mathbb{Q}_{x_0}\left(\int_0^{H_Y(x_0)} f(Y_s)\mathrm{d}s\right), \qquad f \in \mathcal{C}_c,$$

where $H_Y(x) = \inf\{t > 0 : Y_t = x\}$ denotes the first hitting time of x by the process Y, is the unique (up on a constant factor) invariant measure for Y. Further m_0 is absolutely continuous with respect to the Lebesgue measure, with a locally integrable and everywhere positive density given by

$$\frac{q(x_0, y)}{c(y)q(y, x_0)}, \qquad y > 0,$$

where $q(x,y) := \mathbb{Q}_x(H_Y(y) < H_Y(x))$.

(ii) $(Y_t)_{t\geq 0}$ *is positive recurrent if and only if the function* $L_{x,x}$ *has a finite right-derivative at* ρ*, that is,*

$$-L'_{x,x}(\rho) = \mathbb{E}_x\left(H(x)\mathrm{e}^{-\rho H(x)}\mathcal{E}_{H(x)}, H(x) < \infty\right) < \infty \tag{19}$$

for some (and then all) $x > 0$*. In that case, its stationary law, that is* m_0 *normalized to be a probability measure, has the density*

$$\frac{1}{c(y)|L'_{y,y}(\rho)|}, \qquad y > 0.$$

We recall that Lemma 4.1(ii) provides a sufficient condition in terms of the function L_{x_0,x_0} that ensures that (19) holds.

Proof. (i) Indeed, it is well-known that the mean occupation measure of an excursion of Y yields an invariant measure of Y; see, for instance, Getoor [14, §7]. Moreover, since Y is irreducible and recurrent, its invariant measure is unique up to multiplication by a constant; see [19, Theorem 1].

The absolute continuity assertion is deduced from the fact that Y is piecewise deterministic, and more precisely follows the deterministic flow $\mathrm{d}y(t) = c(y(t))\mathrm{d}t$ between its jump times. Specifically, one has then

$$\int_0^{H_Y(x_0)} f(Y_s)\mathrm{d}s = \int_0^\infty f(y)\frac{N(y)}{c(y)}\mathrm{d}y,$$

where $N(y) = \mathrm{Card}\{t \in [0, H_Y(x_0)) : Y_t = y\}$ is the number of visits to y of the excursion of Y away from x_0. In the notation of the statement, it is readily checked that $\mathbb{Q}_{x_0}(N(y)) = q(x_0, y)/q(y, x_0)$, and this yields the expression for the density.

(ii) Using the formula for m_0, the probability tilting, and the martingale property of $\mathcal{M}$, we have

$$\begin{aligned}
\langle m_0, \mathbf{1}\rangle &= \int_0^\infty (1 - \mathbb{Q}_{x_0}(H_Y(x_0) \leq t))\mathrm{d}t \\
&= \int_0^\infty (1 - \mathbb{E}_{x_0}(\mathcal{M}_t, H(x_0) \leq t))\mathrm{d}t \\
&= \int_0^\infty \left(1 - \mathbb{E}_{x_0}\left(\mathcal{M}_{H(x_0)}, H(x_0) \leq t\right)\right)\mathrm{d}t \\
&= \int_0^\infty \mathbb{E}_{x_0}\left(\mathcal{M}_{H(x_0)}, t < H(x_0) < \infty\right)\mathrm{d}t \\
&= \mathbb{E}_{x_0}\left(H(x_0)\mathcal{M}_{H(x_0)}, H(x_0) < \infty\right).
\end{aligned}$$

This proves the first assertion (eventually replacing x_0 by x, which only affects the invariant measure by a constant factor).

The second assertion follows then from uniqueness of the stationary distribution and the fact that the maps $y \mapsto q(x_0, y)$ and $y \mapsto q(y, x_0)$ both have limit 1 as y tends to x_0. This claim can be proved much in the same way as Corollary 4.3, and the full details are left to the reader. ☐

We also point at the following alternative expressions for the occupation measure m_0:

$$\begin{aligned}\langle m_0, f\rangle &= \mathbb{E}_{x_0}\left(\mathrm{e}^{-\rho H(x_0)}\mathcal{E}_{H(x_0)}\int_0^{H(x_0)} f(X_s)\mathrm{d}s, H(x_0)<\infty\right)\\ &= \mathbb{E}_{x_0}\left(\int_0^{H(x_0)} \mathrm{e}^{-\rho s}\mathcal{E}_s\ell(X_s)f(X_s)\mathrm{d}s, H(x_0)<\infty\right),\end{aligned}$$

which follow readily from the probability tilting and the martingale property of $\mathcal{M}$.

We now state our main result about the asymptotic behaviour of growth-fragmentation semigroups.

Theorem 5.3. *Assume that* (15) *and* (19) *hold, so that Y is positive recurrent. Let*

$$\nu(\mathrm{d}y) := \frac{m_0(\mathrm{d}y)}{\bar\ell(y)\langle m_0, \mathbf{1}\rangle} = \frac{\mathrm{d}y}{c(y)\bar\ell(y)|L'_{y,y}(\rho)|}, \qquad y>0.$$

Then for every continuous function f with compact support, we have

$$\lim_{t\to\infty} \mathrm{e}^{-\rho t}T_t f(x) = \bar\ell(x)\int_0^\infty f(y)\nu(\mathrm{d}y).$$

Remark 5.4. We stress that the convergence in Theorem 5.3 can often be significantly strengthened. More precisely, when Y is positive recurrent, it is often possible to show by a classical coupling argument, that the weak convergence

$$\mathbb{Q}_{x_0}(Y_t\in \mathrm{d}y) \Longrightarrow \frac{\mathrm{d}y}{c(y)|L'_{y,y}(\rho)|}$$

actually holds in the total variation sense. Further, when there is a spectral gap, the convergence takes place exponentially fast. See, for instance, [17, 22, 23, 24, 25] for general results in this field. It should be plain from the proof below that these properties can then be transferred to the fragmentation semigroup. We will go into more detail on this topic in the next section, in the special case when the growth rate c is linear.

Proof (of Theorem 5.3). The Feynman-Kac solution to the growth-fragmentation equation given in Lemma 2.2 can be now expressed in terms of $(Y_t)_{t\geq 0}$ as

$$T_t f(x) = \mathrm{e}^{\rho t}\bar\ell(x)\mathbb{Q}_x\left(f(Y_t)/\bar\ell(Y_t)\right).$$

Recall from Corollary 5.2(ii) that Y is positive recurrent whenever (19) holds, and we conclude that

$$\lim_{t\to\infty}\mathrm{e}^{-\rho t}T_t f(x) = \bar\ell(x)\int_0^\infty \frac{f(y)}{\bar\ell(y)}\times\frac{1}{c(y)|L'_{y,y}(\rho)|}\,\mathrm{d}y = \bar\ell(x)\langle\nu, f\rangle.$$

□

Remark 5.5. In the same vein, it might be interesting to point at a similar application of the ratio limit theorem for point recurrent Markov processes (see, for instance, [20, Corollary 20.8] for a statement of this theorem in discrete time) which holds also in the null recurrent case. Specifically, assume (15) holds. Then, for every $f, g \in \mathcal{C}_c$ with $g \geq 0$ and $g \not\equiv 0$, and every $x > 0$, we have

$$\lim_{t\to\infty} \frac{\int_0^t e^{-\rho s} T_s f(x)\,\mathrm{d}s}{\int_0^t e^{-\rho s} T_s g(x)\,\mathrm{d}s} = \frac{\langle m_0, f/\bar{\ell}\rangle}{\langle m_0, g/\bar{\ell}\rangle}.$$

We now conclude this section by observing that the asymptotic profile ν is an eigenmeasure with eigenvalue ρ of the growth-fragmentation operator $\mathcal{A}$, at least under some mild assumptions. In this direction, recall that $\mathcal{A}\bar{f}(x) = x\bar{\mathcal{A}}f(x)$, where $\bar{f}(x) = xf(x)$ and $f \in \mathcal{D}(\bar{\mathcal{A}})$.

Proposition 5.6. *Assume* (19) *holds and that* ℓ *is bounded away from* 0 *on* $(0,\infty)$*. Then* ν *is an eigenmeasure of the dual operator* $\mathcal{A}^*$ *of* $\mathcal{A}$*, with eigenvalue* ρ*, that is* $\langle \nu, \mathcal{A}\bar{f}\rangle = \rho\langle \nu, \bar{f}\rangle$ *for every* $f \in \mathcal{D}(\bar{\mathcal{A}})$.

Proof. Setting $\bar{\nu}(\mathrm{d}y) = y\nu(\mathrm{d}y)$, we need to check that $\langle \bar{\nu}, \bar{\mathcal{A}}f\rangle = \rho\langle \bar{\nu}, f\rangle$ for every function $f \in \mathcal{D}(\bar{\mathcal{A}})$. Because ν is proportional to $m/\bar{\ell}$, it suffices to prove the identity with m/ℓ replacing $\bar{\nu}$. Further, $\bar{\mathcal{A}}f = \mathcal{G}f + \underline{c}f$, where $\mathcal{G}$ is the infinitesimal generator of X. So we have to verify that

$$\langle m/\ell, \mathcal{G}f + \underline{c}f - \rho f\rangle = 0 \qquad \text{for every } f \in \mathcal{D}(\mathcal{G}) = \mathcal{D}(\bar{\mathcal{A}}).$$

That is, using the notation $\mathcal{G}_Y$, defined in (17), for the generator of Y, we must show

$$\langle m, \mathcal{G}_Y(f/\ell)\rangle = 0 \qquad \text{for every } f \in \mathcal{D}(\mathcal{G}). \tag{20}$$

If we set $g = f/\ell$, then the process given earlier in (18) is a $\mathbb{Q}_x$-local martingale. Moreover, it remains so when stopped at $H_Y(x)$. If we assume that ℓ is bounded away from 0 on $(0,\infty)$, then both g and $\mathcal{G}_Y g$ are bounded. Recall further that the occupation measure m_0 of the excursion of Y away from 0 is finite, since thanks to Corollary 5.2, (19) ensures that Y is positive recurrent. We deduce from the optional sampling theorem that

$$\mathbb{Q}_{x_0}\left(\int_0^{H_Y(x_0)} \mathcal{G}_Y g(Y_s)\mathrm{d}s\right) = 0,$$

that is, by definition of m_0, (20) holds. □

For the sake of completeness, we mention the following simple result which ensures that ℓ remains bounded away from 0 on $(0,\infty)$. We omit the proof, since it is a straightforward modification of that of Lemma 4.6.

Lemma 5.7. *Assume that*

$$\liminf_{x\to 0+} \underline{c}(x) > \rho \quad \textit{and} \quad \liminf_{x\to\infty} \underline{c}(x) > \rho.$$

Then $\inf_{(0,\infty)} \ell > 0$.

6 The case of linear growth rate

We shall now discuss in detail the simple case when the function c is linear, namely

$$c(x) = ax\,, \qquad x > 0,$$

for some $a > 0$. We stress that is equivalent to requesting that the identity function is an eigenfunction of $\mathcal{A}$ with eigenvalue a,

We first consider the case in which X is recurrent. Then, $\mathbb{P}_{x_0}(H(x_0) < \infty) = 1$, and we see that (15) holds with $\rho = a$. Hence $\mathcal{E}_t \equiv \mathrm{e}^{at}$ and the semigroup T_t representing the solution to the growth-fragmentation equation (3) is simply given by

$$T_t f(x) = x\mathrm{e}^{at}\mathbb{E}_x[f(X_t)/X_t], \qquad f \in \bar{\mathcal{C}}_c, \quad x > 0.$$

Even more, $\ell(x) \equiv 1$, and the martingale multiplicative functional is trivial, namely $\mathcal{M}_t \equiv 1$, and so we have $Y = X$. As a consequence, if X is also positive recurrent and thus possesses a (unique) stationary distribution, say σ, then we have the convergence

$$\lim_{t\to\infty} e^{-at}T_t f(x) = x\langle \nu, f\rangle, \quad \text{with } \nu(\mathrm{d}y) = y^{-1}\sigma(\mathrm{d}y) \tag{21}$$

for all continuous f with compact support, as we showed in Theorem 5.3.

In this case, the main difficulty is therefore to provide explicit criteria, in terms of k, to ensure that X is positive recurrent, or even exponentially ergodic. There is a wealth of literature concerning such conditions, with the main technique being the application of so-called Foster–Lyapunov criteria. A good introduction to the field may be found in Hairer [17], and the classic monograph of Meyn and Tweedie [22] gives a thorough grounding in the discrete-time setting. The basic notions have been applied and extended many times; as a sample, [25] discusses storage models and queues, [1] looks at the example of kinetic Fokker-Planck equations, and [18] studies stochastic delay equations and the stochastic Navier–Stokes equations.

Recently, Bouguet [6] made a study of the conservative growth-fragmentation equation, which is closely related to our equation (1). Among several interesting results, he studied the asymptotic behaviour of solutions by means of Foster–Lyapunov techniques. Some of the key assumptions in [6] are as follows:

Assumption 6.1. (i) $K(x) > 0$ and $c(x) > 0$ for all $x > 0$.

(ii) There exist constants $\beta_0, \beta_\infty, \gamma_0, \gamma_\infty$ such that

$$K(x) \sim \beta_0 x^{\gamma_0} \text{ as } x \to 0 \qquad \text{and} \qquad K(x) \sim \beta_\infty x^{\gamma_\infty} \text{ as } x \to \infty. \tag{22}$$

(iii) If we define

$$M_x(s) := \frac{1}{K(x)}\int_0^x (y/x)^s\,\bar{k}(x,y)\,\mathrm{d}y \qquad \text{and} \qquad M(s) := \sup_{x>0} M_x(s),$$

then there exist $A > 0$ such that $M(A) < 1$, and $B > 0$ such that $M(-B) < \infty$.

Of course, some restrictions on the exponents in point (ii) are imposed by our assumptions (4) and (11), and these will be made explicit below.

The methods of Bouguet are natural to apply in our situation, and the arguments carry over with minimal modifications. We therefore present in the following result a sufficient criterion for exponential ergodicity, which is the strongest case; weaker assumptions can be made in order to show only ergodicity, and we refer to [6] for more details.

For the result below, recall that by the Riesz representation theorem, for every $x > 0$, there exists a family of measures $(\mu_t^x)_{t\geq 0}$ with the property that $\langle \mu_t^x, f\rangle = T_t f(x)$ for any continuous, compactly supported function $f\colon (0,\infty) \to \mathbb{R}$. Moreover, the measures $yx^{-1}\mathrm{e}^{-at}\mu_t^x(\mathrm{d}y)$ are probability measures. Finally, we recall the definition of the *total variation distance* between two probability measures P and Q on $(0,\infty)$ as being given by

$$\mathrm{d_{TV}}(P,Q) = \frac{1}{2}\sup\{|P(B) - Q(B)| : B \subset (0,\infty), B \text{ Borel set}\}.$$

This discussion permits us to state the following result:

Proposition 6.2. *Suppose $c(x) = ax$ for some $a > 0$ and that Assumption 6.1 is in place. Furthermore, assume that $\gamma_\infty = 0$ and $a/\beta_\infty < (1 - M(A))/A$, and that either $\gamma_0 > 0$ or else $\gamma_0 = 0$ and $a/\beta_0 < (M(-B)-1)/B$. Let $V\colon (0,\infty) \to (0,\infty)$ be a smooth function such that $V(x) = x^{-B}$ for $x \leq 1$ and $V(x) = x^A$ for $x \geq 2$.*

Then, the Markov process X has a unique stationary distribution σ. There exist two constants $\varepsilon > 0$ and $C < \infty$ such that, for every $x > 0$, the semigroup T_t giving the solution of the growth-fragmentation (3) *has the following asymptotic behaviour:*

$$\mathrm{d_{TV}}\left(\mathrm{e}^{-at}\frac{y}{x}\mu_t^x(\mathrm{d}y), \sigma(\mathrm{d}y)\right) \leq C(1 + V(x))\mathrm{e}^{-\varepsilon t}.$$

Proof. We summarise the main points of the proof, which Bouguet [6] gives in greater detail. The idea is to show that the Markov process X is exponentially ergodic, using the results of [25, Theorem 6.1]. Thus, in the terminology of that work, we need to show that compact subsets of $(0,\infty)$ are petite for X, that V is a norm-like function, and that there exist $\alpha, \delta > 0$ such that

$$\mathcal{G}V(x) \leq -\alpha V(x) + \delta. \tag{23}$$

The petiteness of compact sets is shown in [6, p. 6], and requires nothing more than the fact that, on compact subsets of $(0,\infty)$, c is bounded away from zero and infinity and K is bounded away from infinity. The condition that V be norm-like entails that $V(x) \to \infty$ as $x \to 0$ or $x \to \infty$, which is plainly true, as well as that it is in the domain of the generator.

The condition (23) requires the more stringent conditions on the asymptotic exponents and the existence of values A and B. We briefly describe the argument. For $x \geq 2$, we have

$$\mathcal{G}V(x) \leq \Big\{aA - K(x)\Big(1 - M_x(A) - M_x(-B)x^{-(A+B)} - Rx^{-A}\Big)\Big\}V(x),$$

where $R = \min_{x\in[1,2]} V(x) > 0$; and for $x \le 1$, we have

$$\mathcal{G}V(x) \le \Big\{-aB + K(x)\Big(M_x(-B) - 1\Big)\Big\}V(x)$$

In the case $x \ge 2$, the term within braces is equal to $aA - K(x)(1 - M(A) + o(1))$, and as $x \to \infty$, this converges to a negative constant precisely when $a/\beta_\infty < (1 - M(A))/A$. Similarly, in the case $x \le 1$, the term in braces is bounded by a negative constant when x is close enough to zero, provided the conditions of the theorem hold. Since V is bounded on compact subsets of $(0, \infty)$, this implies that (23) holds, and so [25, Theorem 6.1] completes the proof. □

Remark 6.3. The reader who compares our result to [6] will notice that many cases in the latter work are not accommodated by our assumptions. The most significant difference is that, in [6], the fragmentation rate K may be unbounded. Giving a version of Proposition 6.2 in this case would involve only a minor adaptation of the proof, but several earlier results of this work, such as the identification of the eigenmeasure ν of $\mathcal{A}^*$ in Proposition 5.6, would become significantly more difficult. Since our main goal in this article is to point out connections with spectral theory, we prefer not to stray too far from the situation where such results may be proved.

We shall next discuss the situation when X is transient, in which we observe different asymptotic behaviour. In this part, we shall focus on the case where the fragmentation kernel is *homogeneous*, in the sense that

$$\bar{k}(x, y) = y^{-1}\pi(\log(y/x)) \quad \text{for some function } \pi \in L^1_+((-\infty, 0)).$$

Then the operator $\mathcal{G}$ is given by

$$\mathcal{G}f(x) = axf'(x) + \int_0^x (f(y) - f(x))\pi(\log(y/x))y^{-1}\,\mathrm{d}y, \qquad f \in D(\bar{\mathcal{A}}).$$

Our analysis will hinge on the observation that $\mathcal{G}$ can be related to the generator of a Lévy process, as we shall shortly make clear.

The growth-fragmentation equation given by the corresponding operator $\mathcal{A}$ was studied in [16, 11, 4], among others. Indeed, the process X corresponds to the so-called 'tagged fragment' in a random particle model, as we briefly described in [4, §6]. Homogeneous growth-fragmentation equations are often studied via a 'cumulant function' κ, which is defined as follows. For $\theta \in \mathbb{R}$, we define $h_\theta\colon (0, \infty) \to \mathbb{R}$ by $h_\theta(x) = x^\theta$, and then h_θ is an eigenfunction of (an extension of) $\mathcal{A}$ with eigenvalue $\kappa(\theta)$; that is, $\mathcal{A}h_\theta = \kappa(\theta)h_\theta$. The function κ can be given explicitly as

$$\kappa(\theta) = a\theta + \int_0^1 (y^{\theta-1} - 1)\pi(\log y)y^{-1}\,\mathrm{d}y, \qquad \theta \in \mathbb{R},$$

and it is smooth and strictly convex. Our basic assumption, for the remainder of this section, is that there exists some $\theta_0 \neq 1$, lying in the interior of the domain of κ, with the property that $\kappa'(\theta_0) = 0$. Observe that in particular, $\kappa(\theta_0) = \min_{\theta\in\mathbb{R}} \kappa(\theta)$.

We now look more closely at X, and introduce the following auxiliary process, which is a Lévy process; for further background on this class of processes, we refer to [3, 21, 32]. Consider a Lévy process ξ composed of a compound Poisson process with negative jumps plus a drift $a > 0$, and such that ξ has an absolutely continuous Lévy measure with density π. Let ψ represent the Laplace exponent of this Lévy process, which means that $\mathbb{E}[\mathrm{e}^{\theta\xi_t} \mid \xi_0 = 0] = \mathrm{e}^{t\psi(\theta)}$. This function is smooth and strictly convex, with Lévy–Khintchine representation as follows:

$$\psi(\theta) = a\theta + \int_{-\infty}^{0} (\mathrm{e}^{\theta u} - 1)\pi(u)\,\mathrm{d}u = a\theta + \int_0^1 (u^\theta - 1)\pi(\log u)u^{-1}\,\mathrm{d}u, \qquad \theta \in \mathbb{R}.$$

It is related to κ via the equation $\psi(\theta) = \kappa(\theta+1) - \kappa(1)$, from which we see that θ_0 satisfies $\psi'(\theta_0 - 1) = 0$. The existence of θ_0 implies that $\psi'(0) = \mathbb{E}[\xi_1 \mid \xi_0 = 0] \neq 0$, which means that either $\lim_{t\to\infty} \xi_t = \infty$ or $\lim_{t\to\infty} \xi_t = -\infty$. In particular, ξ is a transient process.

By comparing $\mathcal{G}$ with the generator of a Lévy process [32, Theorem 31.5], X may be identified as

$$X_t = \mathrm{e}^{\xi_t}, \qquad t \geq 0,$$

and so X is also transient.

A natural component of our analysis in this situation is the inverse function Φ of ψ, defined by $\Phi(q) = \sup\{\theta \in \mathbb{R} : \psi(\theta) = q\}$. It appears in the following expression, in which $\tau(0) = \inf\{t > 0 : \xi_t = 0\}$:

$$\mathbb{E}[\mathrm{e}^{-q\tau(0)}; \tau(0) < \infty \mid \xi_0 = 0] = 1 - \frac{1}{\Phi'(q)}.$$

This formula can be found, for instance, in Lemma 2(i) of Pardo et al. [28].

From this, we can calculate the spectral radius of the growth-fragmentation equation associated with $\mathcal{G}$. Since the return time of ξ to its starting point is equal to that of X, we calculate, using the inverse function theorem,

$$L_{x_0,x_0}(q) = 1 - \frac{1}{\Phi'(q-a)} = 1 - \psi'(\Phi(q-a)).$$

This implies that $\rho = \kappa(\theta_0) = a + \psi(\theta_0 - 1) < a$, so that contrary to the situation where X is recurrent, here the spectral radius is strictly less than the drift coefficient a.

Moreover,

$$-L'_{x_0,x_0}(q) = \frac{\psi''(\Phi(q-a))}{\psi'(\Phi(q-a))},$$

and as $q \downarrow \rho$, we obtain, by the strict convexity of ψ, that $-L'_{x_0,x_0}(\rho) = \infty$. Thus, we are in a situation where the process Y is *null* recurrent.

We now study the function ℓ in more detail. In order to compute it explicitly, we recall (from [21, §3.3], for instance) that the process $(\mathrm{e}^{(\theta_0-1)\xi_t - t\psi(\theta_0-1)})_{t\geq 0}$ is a non-negative martingale. Since $\lim_{t\to\infty} \xi_t/t = \psi'(0) \neq 0$ almost surely (see [21, Exercise 7.2]), the martingale converges almost surely to 0 as $t \to \infty$. We obtain the following explicit formula for ℓ, applying in the third equality the optional sampling theorem [31, Theorem II.77.5] at $H(\log x_0)$.

$$\begin{aligned}\ell(x) = L_{x,x_0}(\rho) &= \mathbb{E}[\mathrm{e}^{-(\rho-a)H(\log x_0)}; H(\log x_0) < \infty \mid \xi_0 = \log x] \\ &= \mathbb{E}[\mathrm{e}^{(\theta_0-1)\log(x_0)-\psi(\theta_0-1)H(\log x_0)}; H(\log x_0) < \infty \mid \xi_0 = \log x]\mathrm{e}^{-(\theta_0-1)\log(x_0)} \\ &= \mathrm{e}^{(\theta_0-1)(\log x - \log x_0)} \\ &= (x/x_0)^{\theta_0-1}.\end{aligned}$$

Furthermore, we can calculate directly from (17) that the generator of Y is given by

$$\mathcal{G}_Y g(x) = axg'(x) + \int_0^x \Big(g(y) - g(x)\Big)(y/x)^{\theta_0-1}\pi(\log(y/x))\,\frac{\mathrm{d}y}{y}.$$

In other words, we have the representation $Y_t = \exp(\eta_t)$, where η is a Lévy process whose Laplace exponent is given by $\theta \mapsto \psi(\theta+\theta_0-1) - \psi(\theta_0-1)$. This Lévy process has the property that $\mathbb{E}[\eta_1 \mid \eta_0 = 0] = 0$, which implies that η is recurrent (see, for instance, [32, Remark 37.9].)

Finally, we wish to study the asymptotic behaviour of the semigroup T_t, or equivalently, the measures μ_t^x introduced earlier. The semigroup can be identified explicitly in terms of our Lévy process η as:

$$T_t f(x) = \mathrm{e}^{\rho t}\bar{\ell}(x)\mathbb{Q}_x\Big[f(Y_t)/\bar{\ell}(Y_t)\Big] = \mathrm{e}^{\kappa(\theta_0)t}x^{\theta_0}\mathbb{E}\Big[f(\mathrm{e}^{\eta_t})\mathrm{e}^{-\theta_0\eta_t} \mid \eta_0 = \ln x\Big].$$

The asymptotics of this semigroup could be studied using Remark 5.5. However, more precise information can be obtained by applying instead a local central limit theorem for η (see [5, Theorem 8.7.1].) In this way, one recovers the formula

$$T_t f(x) \sim \frac{x^{\theta_0}e^{t\kappa(\theta_0)}}{\sqrt{2\pi t\kappa''(\theta_0))}}\int_0^\infty f(y)y^{-(\theta_0+1)}\,\mathrm{d}y, \text{ as } t \to \infty,$$

for f continuous and compactly supported, which was stated as [4, Corollary 3.4], under different assumptions.

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